Debye potentials for electromagnetic waves in the presence of a charged black hole

1977 J. Phys. A: Math. Gen. 10 L7

(http://iopscience.iop.org/0305-4470/10/1/002)

J. Phys. A: Math. Gen., Vol. 10, No. 1, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

Debye potentials for electromagnetic waves in the presence of a charged black hole

G Stephenson
Department of Mathematics, Imperial College, London, UK

Received 15 November 1976

Abstract. Decoupled equations for the Debye potentials describing the interaction of electromagnetic waves with a charged black hole are solved near the outer horizon and at infinity. The results are valid not just for small charge, as in earlier work, but for any charge e provided $|e|<M$, where M is the mass of the black hole.

Much work has been carried out in recent years on the propagation of electromagnetic waves in strong gravitational fields, motivated largely by the interest in electromagnetic scattering and absorption by black holes (see, for example, Misner *et al* 1970, Ruffini *et al* 1972, Fackerall and Ipser 1972, Breuer *et al* 1973, Mashhoon 1973 and Fabbri 1975). The basic difficulty with such problems is the way in which the Maxwell equations are coupled together when put into a curved background space. In terms of the four-potential of the Maxwell field, the equations are coupled together even in a Schwarzschild space, and it was not until the Newman–Penrose formalism (see Newman and Penrose 1962) was developed that decoupling was achieved. More important, the same technique decoupled the electromagnetic equations in the case of the Kerr space which represents a rotating, uncharged black hole. The Newman–Penrose formalism is clearly a powerful tool in this type of work, but unfortunately does not appear to decouple the Maxwell equations in a Kerr–Newman space describing a charged, rotating black hole, except, as shown by Chitre (1976), when the charge is small. The situation is no different for a charged, but non-rotating black hole as described by the Reissner–Nordström metric. The purpose of this letter is to draw attention to the work of Mo and Papas (1972) who showed how the use of Debye potentials could be extended to a spherically symmetric curved space. Using these potentials, the Maxwell equations in the absence of sources and in a spherically symmetric gravitational field were found to decouple into two basic equations for two scalar (Debye) potentials. The Schwarzschild metric is an obvious example of the type of space considered by Mo and Papas, and Fabbri (1975) has applied their work to the problem of the scattering and absorption of electromagnetic waves by a Schwarzschild black hole. As already remarked, the Newman–Penrose formalism is equally appropriate in the Schwarzschild case since decoupling of the equations can be obtained in this way too. However, the work of Mo and Papas is also applicable to the Reissner–Nordström metric where the Newman–Penrose approach fails unless the charge is small. Taking the Reissner–Nordström metric in the form

$$ds^2=\left(1-\frac{2M}{r}+\frac{e^2}{r^2}\right)dt^2-\left(1-\frac{2M}{r}+\frac{e^2}{r^2}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) \quad (1)$$

where M is the mass of the black hole and e its charge, Mo and Papas show that, in the case of free space ($\mu = \epsilon = 1$ in the usual notation), the two Debye potentials U and V are given by

$$U = V = \frac{h(r)}{r} P_l^m(\cos\theta)\, e^{-i\omega t}\, e^{-im\phi}, \tag{2}$$

where $l \geqslant 1$ and $-|l| \leqslant m \leqslant |l|$, and $h(r)$ satisfies the equation

$$\frac{d}{dr}\left[\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)\frac{dh}{dr}\right] + \left[\omega^2\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} - \frac{l(l+1)}{r^2}\right] h = 0 \tag{3}$$

(ω being the angular frequency of the wave). This equation does not appear to be soluble in terms of known functions, but writing it in normal form by letting $h(r) = u(r)f(r)$, where

$$u(r) = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1/2}, \tag{4}$$

we find that $f(r)$ must satisfy the equation

$$\frac{d^2f}{dr^2} + \left(\frac{\omega^2 r^4}{(r^2 - 2Mr + e^2)^2} - \frac{l(l+1)}{(r^2 - 2Mr + e^2)} + \frac{2Mr^3 - 3r^2(M^2 + e^2) + 6Me^2 r - 2e^4}{r^2(r^2 - 2Mr + e^2)^2}\right) f = 0. \tag{5}$$

Now putting (see Rowan and Stephenson 1977)

$$Mx = r - r_+, \qquad r - r_- = M(x + 2d), \tag{6}$$

where r_+ and r_- are the roots of $r^2 - 2Mr + e^2 = 0$ so that

$$r_+ = M + (M^2 - e^2)^{1/2}, \qquad r_- = M - (M^2 - e^2)^{1/2}, \qquad 2Md = r_+ - r_- = 2(M^2 - e^2)^{1/2}, \tag{7}$$

equation (5) becomes

$$\frac{d^2f}{dx^2} + \left(\frac{\omega^2(x + d + 1)^4 M^2}{x^2(x + 2d)^2} - \frac{l(l+1)}{x(x + 2d)} + \frac{2M^4(x + d + 1)^3 - 3(M^2 + e^2)M^2(x + d + 1)^2 + 6M^2e^2(x + d + 1) - 2e^4}{M^4 x^2(x + 2d)^2(x + d + 1)^2}\right) f = 0. \tag{8}$$

For large x this has the form

$$\frac{d^2f}{dx^2} + \omega^2 M^2 f = 0, \tag{9}$$

which has solutions

$$f = A\cos\omega Mx + B\sin\omega Mx, \tag{10}$$

where A and B are arbitrary constants.

For small x (that is, near $r = r_+$), equation (8) takes the form

$$\frac{d^2f}{dx^2} = \left(\alpha + \frac{\beta}{x} + \frac{\gamma}{x^2}\right) f + O(x)f, \tag{11}$$

where O(x) are terms of order x which tend to zero as $x \to 0$, and the constants α, β and γ are given by (after considerable algebra)

$$\alpha = -\left[\frac{\omega^2 M^2(d+1)^2(11d^2-10d+3)}{4d^4} + \frac{l(l+1)}{4d^2} - \frac{(2d+3)}{16d^4} + \frac{3e^2}{M^2}\left(\frac{-3d^3+13d^2+11d+3}{16d^4(d+1)^3}\right) - \frac{e^4}{8M^4}\left(\frac{23d^2+14d+3}{d^4(d+1)^4}\right)\right], \tag{12}$$

$$\beta = -\left[\frac{\omega^2 M^2(3d-1)(d+1)^3}{4d^3} - \frac{l(l+1)}{2d} + \frac{1}{4d^3} + \frac{3e^2}{4M^2}\left(\frac{d^2-2d-1}{d^3(d+1)^2}\right) + \frac{e^4}{2M^4}\left(\frac{3d+1}{d^3(d+1)^3}\right)\right], \tag{13}$$

$$\gamma = -\left[\frac{\omega^2 M^2(d+1)^4}{4d^2} + \frac{(2d-1)}{4d^2} - \frac{3}{4}\frac{e^2}{M^2}\left(\frac{d-1}{d^2(d+1)}\right) - \frac{e^4}{2M^4 d^2(d+1)^2}\right]. \tag{14}$$

Equation (11) may now be put into Whittaker form by writing $\xi = 2\sqrt{\alpha}x$ to give

$$\frac{d^2 f}{d\xi^2} = \left(\frac{1}{4} + \frac{\beta}{2\sqrt{\alpha}\xi} + \frac{\gamma}{\xi^2}\right) f. \tag{15}$$

This equation has Whittaker function solutions (see Whittaker and Watson 1927)

$$f(\xi) = CM_{\kappa,p}(\xi) + DM_{\kappa,-p}(\xi), \tag{16}$$

where

$$\kappa = -\beta/2\sqrt{\alpha}, \qquad p = (\gamma + \tfrac{1}{4})^{1/2}, \tag{17}$$

(provided $2p$ is non-integral) and C and D are arbitrary constants. (Similar results involving Whittaker functions have recently been obtained by Rowan and Stephenson (1976, 1977) in the solution of the radial equation for a massive scalar meson field in a curved background space.)

Using these results we can now write down, from (2), the explicit forms of the Debye potentials in the two regions:

(1) *Large r*

$$U = V = (r^2 - 2Mr + e^2)^{-1/2}(A \cos\omega(r-r_+) + B \sin\omega(r-r_+))P_l^m(\cos\theta)\, e^{-i\omega t}\, e^{-im\phi} \tag{18}$$

$$\sim r^{-1}(A' \cos\omega r + B' \sin\omega r)P_l^m(\cos\theta)\, e^{-i\omega t}\, e^{-im\phi}. \tag{19}$$

(2) *Near* $r = r_+$

$$U = V = (r^2 - 2Mr + e^2)^{-1/2}\left[CM_{\kappa,p}\left(\frac{2\sqrt{\alpha}}{M}(r-r_+)\right) + DM_{\kappa,-p}\left(\frac{2\sqrt{\alpha}}{M}(r-r_+)\right)\right] \times P_l^m(\cos\theta)\, e^{-i\omega t}\, e^{-im\phi} \tag{20}$$

$$\sim (r-r_+)^{-1/2}\left[C'M_{\kappa,p}\left(\frac{2\sqrt{\alpha}}{M}(r-r_+)\right) + D'M_{\kappa,-p}\left(\frac{2\sqrt{\alpha}}{M}(r-r_+)\right)\right] \times P_l^m(\cos\theta)\, e^{-i\omega t}\, e^{-im\phi}, \tag{21}$$

where κ and p are given by equation (17). In all this work there is no restriction on the charge e other than the usual condition $|e| < M$ (see equation (7)).

I am very grateful to D J Rowan for useful discussions during the preparation of this paper. I am also grateful to Professor R J Elliott for the hospitality shown to me during my stay in the Department of Theoretical Physics, University of Oxford, where this work was carried out.

References

Breuer R A, Ruffini R, Tiomno J and Vishveshwara C V 1973 *Phys. Rev.* D **7** 1002
Chitre D M 1976 *Phys. Rev.* D **13** 2713
Fabbri R 1975 *Phys. Rev.* D **12** 933
Fackerall E D and Ipser J R 1972 *Phys. Rev.* D **5** 2455
Mashhoon B 1973 *Phys. Rev.* D **7** 2807
Misner C W, Thorne J and Wheeler J A 1970 *Gravitation* (San Francisco: Freeman)
Mo T C and Papas C H 1972 *Phys. Rev.* D **6** 2071
Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566
Rowan D J and Stephenson G 1976 *J. Phys. A: Math. Gen.* **9** 1631
—— 1977 *J. Phys. A: Math. Gen.* **10** 15–23
Ruffini R, Tiomno J and Vishveshwara C V 1972 *Nuovo Cim. Lett.* **3** 211
Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)